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ON THE GROUPS WHICH ARE DETERMINED BY A GIVEN GROUP.

By DR. G. A. MILLER.

A group (G) generally determines a number of different groups which throw much light on the structure of G. It is the object of this paper to study some of the properties and uses of these groups. When we transform each of the operators of G by any one of them the consequent permutation of these operators determines a substitution. If we use all the operators of any subgroup of G as transformers the corresponding substitutions will evidently form a group which has a 1, α isomorphism to the given subgroup, where the value of α generally depends upon the particular subgroup whose operators have been employed as transformers.

If we employ all the operators of G as transformers the corresponding substitutions form a group which has a 1, β isomorphism to G. This group has been called the *group of cogredient isomorphisms* of G. It is clear that β is the maximum value of α. Hence all the α's are equal to unity whenever $\beta=1$. In this case all the isomorphisms are said to be simple and this process does not lead to any group except those that are found in G.

When some of these isomorphisms are not simple the given process may lead to groups that are not contained in G. The group of lowest order in which this happens is the well known quaternion group of order 8. In this case the group of cogredient isomorphisms is the four-group, which is clearly not contained in the quaternion group. The groups which correspond to the transformations obtained by using the operators of a subgroup are clearly contained in the quaternion group. When the order of such a subgroup exceeds unity $\alpha=\beta=2$.

Since β operators of G must transform its operators according to the same substitution in its group of cogredient isomorphisms, G must contain just β operators that are commutative to each one of its operators. These constitute an Abelian characteristic* subgroup of G. This subgroup may be called the cogredient subgroup of G. Hence the factors of composition of G are the prime factors of β together with the factors of composition of its group of cogredient isomorphisms. The group of cogredient isomorphisms of the group of cogredient isomorphisms is called the second group of cogredient isomorphisms. From the preceding paragraph it follows that the second group of cogredient isomorphisms of the quaternion group is identity.

From the second group of cogredient isomorphisms we may obtain the third in the same manner as the second was obtained from the first, etc. If we arrive at identity by finding the successive groups of cogredient isomorphisms of a given group the group must be solvable since its factors of composition must be prime numbers. There are, however, many solvable groups which do not possess this property, *e. g.* the symmetric group of order 6. Hence the given condition is sufficient but not necessary for the solvability of a group.

G may have simple isomorphisms to itself which cannot be obtained by transforming it by its own operators. All such isomorphisms can be obtained by transforming G by operators that transform it into itself† and they correspond to a group known as the *group of isomorphisms* of G. This group contains the group of cogredient isomorphisms as a selfconjugate (not necessarily characteristic) subgroup. While the group of cogredient isomorphisms of G has a 1, β isomorphism to G the group of isomorphisms of G need not possess such a property; *e. g.* the group of isomorphisms of the four-group is the symmetric group of order 6.

The group of isomorphisms of G is very useful in the study of groups which contain G as a self-conjugate subgroup. Such a group must transform the operators of G according to its group of isomorphisms or according to some subgroup of this group. In substitution groups the group of isomorphisms is very useful to determine the number of intransitive groups that can be formed by making a group simply isomorphic to itself. Among the types of groups whose groups of isomorphisms have received considerable attention are the Abelian groups which contain no operator, besides identity, whose order differs from a given prime number,‡ the alternating and the symmetric group of any degree,‖ and the cyclical group of any order.§

We sometimes arrive at additional groups by finding the successive groups of isomorphisms of G, *i. e.* the group of isomorphisms of the group of isomorphisms, etc. In particular, it is well known that the group of isomorphisms of the group generated by three independent commutative operators of order 2 is the simple group of order 168 and that its second group of isomorphisms is the

*Frobenius, *Berliner Sitzungsberichte*, 1895, page 183.
†Cf. Frobenius, *Berliner Sitzungsberichte*, 1895, page 184.
‡Cf. Moore, *Bulletin of the American Mathematical Society*, vol. 2, 1895, pages 33-43.
‖Cf. Hoelder, *Mathematische Annalen*, vol. 46, 1895, pages 333-345.
§Cf. Burnside's *Theory of Groups*, 1897, page 240.

group of order 336, which may be represented as a transitive group of degree 8. We do not obtain any additional group by finding the higher groups of isomorphisms since the group of isomorphisms of a simple group of composite order is necessarily complete.*

It may happen that a group is simply isomorphic to its group of cogredient isomorphisms as well as to its group of isomorphisms. Such a group has been called a complete group. The symmetric group of every degree except two and six is complete. The group of degree 4 and order 8 is an example of a group that is simply isomorphic to its group of isomorphisms without being simply isomorphic to its group of cogredient isomorphisms, while the alternating group of every degree except three is simply isomorphic to its group of cogredient isomorphisms without being simply isomorphic to its group of isomorphisms. The cyclical group of composite order is clearly not simply isomorphic to either of these groups of isomorphisms.

Every selfconjugate subgroup of a group may be regarded as a modulus with respect to which all of its operators may be divided into sets containing an equal number of operators. These sets of operators determine a group known as the quotient group of the given group with respect to the particular selfconjugate subgroup. We have seen that such a quotient group need not be simply isomorphic to any subgroup of the given group.

If s and t represent any two operators of the group (G), $s^{-1}t^{-1}st$ will also represent an operator of G. All the operators of G that can be represented in the form $s^{-1}t^{-1}st$ generate the commutator subgroup of G. This subgroup has also been called the first derivative of G. With respect to this derivative G is isomorphic to an Abelian group of maximum order. The necessary and sufficient condition that G is solvable is that we arrive at identity when we form its successive derivatives.† G may have several self-conjugate subgroups as well as several characteristic subgroups, but it can have only one commutator subgroup as well as only one cogredient subgroup. If a group coincides with its commutator subgroup it is said to be perfect.

If G is represented as a regular group the largest substitution group of the same elements that transforms G into itself determines the holomorph of G,‡ *i. e.* this substitution group is simply isomorphic to the holomorph of G. The subgroup which includes all the substitutions that do not contain any one element of this substitution group determines the group of isomorphisms of G. The holomorph of G may also be defined as the largest group that transforms G into itself and contains only as many operators as are contained in G that are commutative to every operator of G. When G is a complete group the order of its holomorph is the square of the order of G.

When G contains a subgroup that does not include any selfconjugate subgroup besides identity, it can be represented as a transitive substitution group

*Cf. Ibid, page 238.

†Miller, *American Journal of Mathematics*, vol. 20, 1898, page 277.

‡Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

whose degree is obtained by dividing the order of G by the order of this subgroup. When the subgroup is a maximum subgroup the corresponding transitive group is primitive; when this condition is not satisfied it is imprimitive*. All subgroups of this type that can be made to correspond in a simple isomorphism of G to itself, and only these, lead to the same transitive group. Hence we can readily obtain all the transitive substitution groups that are determined by G, *i. e.* those which are simply isomorphic to G.

From what precedes it follows that the complete study of a group (G) implies the study of its successive groups of isomorphisms and those of cogredient isomorphisms, its selfconjugate subgroups and the corresponding quotient groups, its successive derivatives and the corresponding solvable groups, its characteristic and cogredient subgroups, its subgroups that do not involve any self-conjugate subgroup besides identity and the corresponding transitive substitution groups, etc. From this standpoint each group generally determines a group complex whose various parts throw much light on the structure of the group.

Chicago, August, 1898.

*Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

INFINITY, THE INFINITESIMAL, AND ZERO.

By HENRY HEATON, M. Sc., Atlantic, Iowa.

So much has been written upon this subject that I do not flatter myself that I can write anything new. I shall only attempt to point out a few of the things that have been written that to my mind plainly cannot be true, and with these errors in view endeavor to express the truth as I see it. If in so doing I fall into error I shall have the satisfaction of knowing that greater men have done the same.

In the *Analyst*, Vol. VIII., pages 105-113, Professor Judson has a very interesting article upon this subject in which he quotes freely from many distinguished authors many things that are plainly fallacious. Yet when he attempts to outline his view of the subject it seems to the writer that he blunders fully as badly as those whom he criticises.

He says: "(7) *A variable which decreases indefinitely and which by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinitesimal. Thus, read when $x=\circ$, when x decreases indefinitely or when x is an infinitesimal.

(8) If a be a constant the expression $a/\circ = \infty$ and $a/\infty = \circ$ are rigidly exact."

He says further, "$\infty \pm a = \infty$, also $\circ \times \infty = 0/0$ is wholly indeterminate. We cannot write $a \pm \circ = a$, nor can we write $a \pm \circ = c$, if c is a constant."

He quotes Professor DeMorgan with evident approval as saying that he dates his first clear conception of mathematical infinity from the time when he rejected the relation $a/0 = \infty$.

He says: "Instead of saying the 'tangent is infinite when $x = 90°$' we may say 'the tangent of x becomes infinite as x approaches $90°$'. In tracing curves, if $y = x^3/(a-x)$, we should say, y becomes infinite as x approaches a, and not when $x = a$."

Dr. Davies, in his Differential Calculus, Art. 56. says: "When therefore each part becomes *infinitely small*, any *finite* number of them is 0, but an infinite number of them is equal to a finite quantity."

If a *finite* number of them equals 0, how about *one* of them?

In Lilley's Higher Algebra under the interpretation of $a/0$ I find the following remarkable statement. "Dividing 12 by a number that decreases by 1 each time commencing with +4, we have $\frac{12}{+4} = 3$, $\frac{12}{+3} = 4$, $\frac{12}{+2} = 6$, $\frac{12}{+1} = 12$ $\frac{12}{0} = 0$, etc. 3 means that 3 times +4 can be subtracted from 12 and leave 0; 4 means that 4 times +3 can be subtracted from 12 and leave 0; the quotient 0 means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and *leave zero*." His general conclusion is then that $a/0 = 0$.

He says: "If a constant be divided by an infinitesimal, the quotient is infinity, and if a constant be divided by infinity, the quotient is infinitesimal." Thus agreeing with Judson.

His conclusion that $a/0 = 0$ is the result of a mere play upon words. He overlooks the fact that if $a/0 = 0$ then if 0 be subtracted from a 0 times the result must be 0.

I am willing to agree with these writers that $a/0$ is the symbol of impossibility; but I would say that ∞ is also the symbol of impossibility. Hence, $a/0 = \infty$.

a/x does not equal infinity so long as x has the slightest shadow of value.

I believe that mathematical infinity is something beyond the ken of even a Professor DeMorgan, that he was never so far from having a correct conception of it as when he claimed to have a clear conception of it. Tan$x$ does not become infinite as x approaches $90°$, for so long as x differs by the shadow of a hair from $90°$, tan$x$ is finite.

I think I know exactly what is meant by the term zero. But I can have no conception either of infinity or of the infinitesimal, and I think it would be well if mathematicians would let both pretty severely alone.

If $u = x^2$ and du = an increment that u receives as a result of an increment dx to the value of x; then $du = 2xdx + dx^2$, and $\frac{du}{dx} = 2x + dx$. This equation cannot reduce to $\frac{du}{dx} = 2x$ in any other way than by putting dx, and consequently

du=absolute zero. If dx has any value at all, infinitesimal or otherwise, $\frac{du}{dx}$ does not equal $2x$.

If $u=x^n$, the general expression for $\frac{du}{dx}$, supposing du and dx to have definite values, is $\frac{du}{dx}=nx^{(n-1)}+\frac{n(n-1)}{1.2}x^{(n-2)}dx+\frac{n(n-1)(n-2)}{1.2.3}x^{(n-3)}dx^2$, etc. When $dx=0$, and not before, this reduces to $\frac{du}{dx}=nx^{(n-1)}$. This is called the differential coefficient of u with respect to x.

It may be ridiculous to say 0/0 equals anything. But there can certainly be no objection to saying that the differential coefficient of a function u with respect to x, is the value to which the *general expression for* $\frac{du}{dx}$ *reduces* when dx is equated to 0, for this is what the differential coefficient really is.

It may be readily shown that the differential coefficients, or derivatives, of two equal functions are always equal, and that if two derivatives are equal *their primitives can differ only by a constant quantity.*

The results obtained by integration are correct because of this last truth.

The process called integration, instead of being the summing of an infinite number of infinitely small quantities, is really the process of finding the primitive corresponding to a given derivative.

As an illustration I shall compute the variable volume, V, of a right cone whose variable altitude is x and the radius of whose variable base is $\frac{ax}{h}$. If dV is any increment which V receives as a result of an increment dx to x, then $dV=\frac{a^2x^2\pi}{h^2}dx+p(dx^2)$. Where p is a variable finite quantity known to be less than the circumference whose radius is $\frac{ax}{h}$. Hence $\frac{dV}{dx}=\frac{a^2x^2\pi}{h^2}+pdx$. When dx is equated to zero the expression for $\frac{dV}{dx}$ reduces to $\frac{a^2x^2\pi}{h^2}$. Hence, $\frac{a^2x^2\pi}{h^2}$ is the derivative of V with respect to x. But $\frac{a^2x^2\pi}{h^2}$ is the derivative of $\frac{a^2x^3\pi}{3h^2}+C$. Hence $V_0{}^r=\frac{a^2h\pi}{3}$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

98. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A poor man borrowed \$20 which he repaid in eleven monthly installments of \$2 each; what was the annual rate of interest (reckoned as simple interest)?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Natural Science, Chester High School, Chester, Pa.

Here we use the same formula we have used often before in the MONTHLY:

$$p=\frac{Pr(1+r)^n}{(1+r)^n-1}.$$

Where $p=2$, $P=20$, $n=11$.

$\therefore 2(1+r)^{11}-2=20r(1+r)^{11}$. $\therefore 2(1+r)^{11}(1-10r)=2$.

$\therefore r=.016$ nearly. $12r=.192=19\frac{1}{5}\%$ nearly.

$\therefore 19\frac{1}{5}\%=$rate of interest.

Also solved by *ELMER SCHUYLER*.

99. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

If 300 cats kill 300 rats in 300 minutes, how many cats will kill 100 rats in 100 minutes?

I. Solution by the PROPOSER.

1. If 300 cats kill 300 rats in 300 minutes,
2. 1 cat will kill 1 rat in 300 minutes,
3. 1 cat will kill 100 rats in 3000 minutes, and
4. 300 cats will kill 100 rats in 100 minutes.

II. Solution by CHARLES C. CROSS, Libertytown, Md.

If 300 cats catch 300 rats in 300 minutes, then 1 cat will catch 300 rats in 9000 minutes, or 1 cat will catch 100 rats in 3000 minutes.

Hence 300 cats will catch 100 rats in 100 minutes.

Also solved by *G. B. M. ZERR*, *FREMONT CRANE*, and *ALOIS F. KAVORIK*.

ALGEBRA.

86. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, and Counsellor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+yz=16$.....(A); $y^2+xz=17$.....(B); $z^2+xy=22$.....(C), for all the roots.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Chester High School, Chester, Pa., and Prof. J. SCHEFFER, A. M., Hagerstown, Md.

Let $y=vx$, $z=wx$.

$\therefore x^2+vwx^2=16$, $v^2x^2+wx^2=17$, $w^2x^2+vx^2=22$.....$(1, 2, 3)$.

(1)÷(2), $(1+vw)\div(v^2+w)=\frac{6}{7}$.....(4).

(1)÷(3), $(1+vw)\div(w^2+v)=\frac{8}{11}$.....(5).

Eliminating v between (4) and (5), we get

$59w^4-378w^3+704w^2-315w-106=0$.....(6).

$\therefore$ $w=2$, or $-.21774$, or $2.81226\pm.17307\sqrt{(-1)}$.

$v=1.5$, or 1.02072, or $2.29118\pm.14936\sqrt{(-1)}$.

$x=2$, or 4.53566, or

$y=3$, or 4.62964,

$z=4$, or $-.98759$.

v is found from (5). $\therefore$ $v=(8w^2-11)\div(11w-8)$.

The imaginary values of x, y, and z can be found from $x=4\div\sqrt{(1+vw)}$, $y=vx$, $z=wx$.

[The only rational value of (6) is $w=2$. $\therefore$ $v=\frac{3}{2}$, and substituting $z=2x$, $y=3x\div2$ in the first of the given equations we get $4x^2=16$, whence $x=\pm2$, $y=\pm3$, $z=\pm4$. Dividing the biquadratic by $w-2$, we get the cubic equation $59w^3-260w^2+184w+73=0$, which furnishes the other roots of w, these being, however, irrational. SCHEFFER.]

II. Solution by the PROPOSER.

By r (ratio) $y=rx$: ..... $z^2+rx^2=22$.....(C).

$x^2+rxz=16$.....(A); deduct $r(B)-(A)$.

$\therefore$ $r^2x^2+xz=17$.....(B); $(r^3-1)x^2=17r-15$.....(D).

Deduct $r^2(A)-(B)$ ....$(r^3-1)xz=16r^2-17$.....(E).

$(r^3-1)(C)$ is.....$(r^4-r)x^2+(r^3-1)z^2=22r^3-22$.

$r'(D)$ is.....$(r^4-r)x^2$.....$=17r^2-16r$. Subtracting,

$(r^3-1)z^2=22r^3-17r^2+16r-22$.

Multiplying this equation by (D),

$(r^3-1)^2x^2z^2=374r^3-641r^3+544r^2-630r+352$.

$(r^3-1)^2x^2z^2=256r^4-544r^2+289$. Subtracting $(E)^2$,

$0=118r^4-641r^3+1088r^2-630r+63$ .....$(E)^a$

Obviously $r=1.5$. $\therefore$ $177-696\quad588-63$

$\therefore$ $118r^3-464r^2+392r-42=0$.....(E_1).

$\frac{1}{2}(E_1)$ is.....$59r^3-232r^2+196r-21=0$.....(F) $\therefore$ $r=1.5$.

$r_1=0.125069\ 382246\ 530323\ 732\pm$

$r_2=1.021700\ 042125\ 541767\ 819\pm$

$r_a=2.785433\ 965458\ 436383\ 025$. *Same way*, $z=tx$.

$\therefore$ $59t^3-260t^2+184t+53=0$.....(G). *Obviously*, $t=2$.

$t_1=\ 1.207289\ 829921\ 583026\ 2730\pm$

$t_2=-0.217739\ 956753\ 32493$ .....

$t_a=\ 3.417229\ 787848\ 69105$..... *Also got by*

$(E)\div(D)=t=(16r^2-17)\div(17r-16)$..... (H) .....is $[16(1.5)^2-17]\div[17(1.5)-16]=2=t$, etc.

Get x by (A).....$x^2(1+rt)=16$; *obviously* $rt=\frac{3}{2}(2)=3$.

$\therefore x^2(1+3)=16$; .....$x=\pm_1 2$; $y=\pm_1 3$; $z=\pm_1 4$.

Test r_2.....t_1, etc., by $(B)(C)$ gives $17(t^2+r)=22(r^2+t)$.....(J).

Thus $r_1=0.125$.....in (J) *fixes* $t_1=1.2073$ (4 true decimals).

$r_2=1.0217$ in (J) *fixes* $t_2=0.217739$ (6 true decimals).

$r_a=2.785434$ in (J) *fixes* $t_a=3.417229$ (6 true decimals).

$\therefore x_1=\pm_1$.....to	$y_1=\pm_1$.....to	$z_1=\pm_1$ .....
3.728406 657543 8	0.466309 516672 3	4.501267 432220 0
$x_2=\pm_1$.....to	$y_2=\pm_1$.....to	$z_2=\pm_1$.....
4.536281 490681 969	4.634718 990123 0830	0.987729 704967 0174
$x_a=\pm_1$.....to	$y_a=\pm_1$.....to	$z_a=\pm_1$.....
1.233342 644210 0772	3.435894 472231.4779	4.214615 222418.7456

Read $\pm_1$.....$\pm_1$.....$\pm_1$ "change in unison," not $\pm$ *prime*, $\mp$ *prime*. One page finds *all* 24 roots of x, y, z. Q. V. D.

[*Prof. Charles C. Cross* remarks that two solutions are given in the *Mathematical Diary* for 1831, pages 150-151, and that there is quite an interesting history of this "curious and important question" given by "Unicorn," of North Carolina.

87. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

A starts to travel around a circular island at a given point and travels at the rate of 5 miles in 4 hours. One half hour after *A*, *B* starts from a point directly opposite from *A* and travels in an opposite direction at the rate of 4 miles in 3 hours. One hour afterwards *C* starts from the same point as *A* and travels in an opposite direction to *A* at the rate of 3 miles in 2 hours. One half hour afterwards *D* starts from the same point as *B* and travels in an opposite direction to *B* at the rate of 2 miles in 1 hour. Required the size of the island, and when they will all be together, and how far each will have traveled at the accomplishment of this event.

Solution by the PROPOSER.

This problem furnishes a good example in indeterminate analysis. Of course an unlimited number of answers may be obtained each of which will satisfy the conditions of the problem ; but we shall be content to show in what manner one set may be obtained.

A's rate of travel equals $1\frac{1}{4}$ miles per hour, *B*'s rate equals $1\frac{1}{3}$ miles per hour, *C*'s rate equals $1\frac{1}{2}$ miles per hour, and *D*'s rate equals 2 miles per hour. *B* starts $\frac{1}{2}$ hour after *A* starts, *C* starts $1\frac{1}{2}$ hours after *A*, and *D* starts 2 hours after *A*.

Let c=the circumference of the island; then *A* and *B* will be together for first time after *A* starts in $[(\frac{1}{2}c-1\frac{1}{4}\times\frac{1}{2})/(1\frac{1}{4}+1\frac{1}{3})]+\frac{1}{2}=(6c+8)/31$ hours.

For the second time in $[(1\frac{1}{2}c-1\frac{1}{4}\times\frac{1}{2})/(1\frac{1}{4}+1\frac{1}{3})]+\frac{1}{2}=(18c+8)/31$ hours ; and for the $(m+1)$th time in $[(6+12m)c+8]/31$ hours.....(A).

A and *C* will be together for the first time after *A* starts in $[(c-1\frac{1}{2}\times1\frac{1}{4})/(1\frac{1}{2}+1\frac{1}{4})]+1\frac{1}{2}=(4c+9)/11$ hours.

For the second time in $[(2c-1\frac{1}{2}\times1\frac{1}{4})/(1\frac{1}{2}+1\frac{1}{4})]+1\frac{1}{2}=(8c+9)/11$ hours ; and for the $(n+1)$th time in $[(4+4n)c+9]/11$ hours.....(B).

A and D will be together for the first time after A starts in $[(\frac{1}{2}c+1\frac{1}{4}\times 2)/(2-1\frac{1}{4})]+2=(2c+16)/3$ hours.

For the second time in $[(1\frac{1}{2}c+1\frac{1}{4}\times 2)/(2-1\frac{1}{4})]+2=(6c+16)/3$ hours, and for the $p+1$ time in $[(2+4p)c+16]/3$ hours .....(C).

Now when the four persons are all together, the general expressions (A), (B), and (C) will represent the same number of hours. Hence, at the occurrence of such an event, we may equate them, and we shall have

$[(6+12m)c+8]/32=[(4+4n)c+9]/11$.....(1), and

$[(4+4n)c+9]/11=[(2+4p)c+16]/3$.....(2).

From equation (1) we obtain $c=191/(132m-124n-68)$.....(3).

From equation (2) we obtain $c=149/(12n-44p-10)$.....(4).

Equating (3) and (4), clearing of fractions, and dividing the resulting equations by 44, we obtain $447m-472n+191p=153$ .....(5).

Here we have three unknown quantities to be determined from a single equation, but as their values are necessarily restricted to whole numbers, we determine one set of such values as follows: Give to one of the unknown quantities any integral value we choose; as 8 is a convenient value, put $p=8$, and (5) will become $447m-472n=-1375$.....(6), or $m=n+(25n-1375)/447$.....(7). Now as the fractional expression $(25n-1375]/447$ must be a whole number, in order to make m such, give to n any integral value in it that shall render the expression a positive quantity. For instance, put $n=70$ and the expression will become $\frac{3}{4}\frac{5}{4}\frac{5}{7}$. We perceive that n must be increased to give an integral result, and by how much we determine as follows:

$(25\times 447-375)/25=332$. $\therefore$ if we take $n=432+70=502$ we shall find that the expression will equal 25, a whole number, and by substitution in (7) we shall find $m=502+25=527$. $\therefore$ the values $p=8$, $m=527$, and $n=502$ will satisfy equation (5). Substituting these values in equations (3) or (4), we have $c=\frac{1}{38}$ of a mile, the circumference of the island. Again, substituting these values found for m, n, p, and c in either of the expressions (A), (B), or (C), we find that A, B, C, and D will be altogether in $5\frac{2}{19}$ hours after A starts.

Proof and remaining answers. In $5\frac{2}{19}$ hours A will travel $\frac{107}{19}\times\frac{5}{4}=7\frac{3}{76}$ miles, and he will go around the island $7\frac{3}{76}\div\frac{1}{38}=262\frac{1}{2}$ times, and hence will be at B's starting point at the end of the time. B, who travels $\frac{1}{2}$ hour less than A, or $5\frac{2}{19}-\frac{1}{2}=\frac{195}{38}$ hours, will travel $\frac{195}{38}\times\frac{4}{3}=6\frac{6}{19}$ miles, and he will go around the island $6\frac{6}{19}\div\frac{1}{38}=260$ times, and hence will be at his starting point at the end of the time. C, who travels $1\frac{1}{2}$ hours less than A, or $5\frac{2}{19}-1\frac{1}{2}=\frac{157}{38}$ hours, will travel $\frac{157}{38}\times\frac{3}{2}=6\frac{5}{76}$ miles, and he will go around the island $6\frac{5}{76}\div\frac{1}{38}=235\frac{1}{2}$ times, and hence will be at B's starting point at the end of the time. D, who travels 2 hours less than A, or $5\frac{2}{19}-2=3\frac{2}{19}$ hours, will travel $3\frac{2}{19}\times 2=7\frac{5}{19}$ miles, and he will go around the island $7\frac{5}{19}\div\frac{1}{38}=276$ times, and he will be at B's starting point at the end of the time. As they will all be at B's starting point at the end of $5\frac{2}{19}$ hours after A starts, they must all be together in that time.

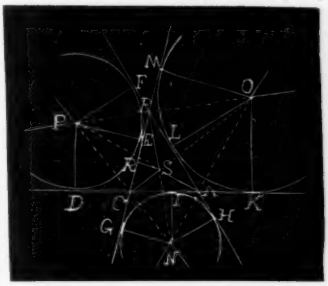
GEOMETRY.

97. Proposed by CHARLES C. CROSS, Libertytown, Md.

Prove by pure geometry: The radius of a circle drawn through the centers of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let ABC=any triangle, S be the center of the inscribed circle, and P, N, and O, the centers of the escribed circles. Put $BC=a$, $AC=b$, $AB=c$, $SR=r$, $PE=r_1$, $NH=r_2$, $OM=r_3$, $\frac{a+b+c}{2}=s$, $\triangle$=area of $\triangle ABC$.



$\triangle ABC=\triangle BCS+\triangle ACS+\triangle ABS$

$=\frac{ra}{2}+\frac{rb}{2}+\frac{rc}{2}=\frac{1}{2}r(a+b+c)=rs$ ..... (1).

Polygon $ACPB=\triangle ABC=\triangle CPB=\triangle ACP+\triangle ABP$; or $rs+\frac{r_1a}{2}=\frac{r_1b}{2}=\frac{r_1c}{2}$; whence $rs=r_1\left(\frac{a+b+c}{2}-a\right)$, or $rs=r_1(s-a)$ ..... (2).

Polygon $BANC=\triangle ABC+\triangle ANC=\triangle BAN+\triangle BCN$; or $rs+\frac{r_2b}{2}=\frac{r_2c}{2}=\frac{r_2a}{2}$; whence $rs=r_2\left(\frac{a+b+c}{2}-b\right)$, or $rs=r_2(s-b)$ ..... (3).

Similarly, we find $rs=r_3\left(\frac{a+b+c}{2}-c\right)$, or $rs=r_3(s-c)$ ..... (4).

Then (1), (2), (3), and (4), respectively, give $rs=\triangle$, $r_1(s-a)=\triangle$, $r_2(s-b)=\triangle$, and $r_3(s-c)=\triangle$. Whence $r=\frac{\triangle}{s}$, $r_1=\frac{\triangle}{s-a}$, $r_2=\frac{\triangle}{s-b}$, and $r_3=\frac{\triangle}{s-c}$.

As it is readily proved, we simply state that the lines drawn from the centers of the escribed circles, respectively, to the vertices of the opposite angles of the triangle, are (1) The bisectors respectively of the angles of the triangle, and hence pass through the center of the inscribed circle, and (2) The perpendiculars respectively to the lines joining the centers of the escribed circles; as BN bisects $\angle ABC$ and is perpendicular to PO.

Again, leaving the burden of proof to the reader, hinting, however, that $\angle MBL$ is bisected by PO as a property of escribed circles, we find the following sets of triangles similar: PEB and OLB; OLA and NIA; NIC and PEC.

Let $EC=x$, $IC=y$, $AL=z$; then $BE=a-x$, $AI=b-y$, $BL=c-z$.

From the sets of similar triangles just noted we obtain the following proportions, remembering that $PE=r_1$, $NI=r_2$, and $OL=r_3$.

$c-z : r_3 = a-x : r_1$, or $r_1(c-z) = r_3(a-x)$ .....(1).

$z : r_3 = b-y : r_2$, or $z = \frac{r_3(b-y)}{r_2}$ .....(2).

$y : r_2 = x : r_1$, or $x = \frac{r_1 y}{r_2}$ .....(3).

Substituting the values of z and x, of (2) and (3), in (1), we find

$$y = \tfrac{1}{2}b + \frac{r_2 a}{2r_1} - \frac{r_2 c}{2r_3} = \tfrac{1}{2}b + \frac{a(s-a)}{2(s-b)} - \frac{c(s-c)}{2(s-b)} = s-a.$$

Whence $b-y=s-c$, $x=s-b$, $a-x=s-c$, $z=s-b$, and $c-z=s-a$.

Then $PB = \sqrt{[(a-x)^2 + r_1^2]} = \sqrt{\left(\frac{ac(s-c)}{s-a}\right)}$; $BO = \sqrt{\left(\frac{ac(s-a)}{s-c}\right)}$; $PO = PB + BO = \frac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$.

Also $AO = \sqrt{\left(\frac{bc(s-b)}{s-c}\right)}$; $AN = \sqrt{\left(\frac{bc(s-c)}{s-b}\right)}$; $NO = AO + AN = \frac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$.

Also $NC = \sqrt{\left(\frac{ab(s-a)}{s-b}\right)}$; $PC = \sqrt{\left(\frac{ab(s-b)}{s-a}\right)}$; $NP = NC + PC = \frac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$.

Also $NB = \sqrt{\left(\frac{acs}{s-b}\right)}$; $PA = \sqrt{\left(\frac{bcs}{s-a}\right)}$; $OC = \sqrt{\left(\frac{abs}{s-c}\right)}$.

From the similar triangles PAN and PCS, we have

$$PN : PA = PS : PC, \text{ or } \frac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}} : \sqrt{\left(\frac{bcs}{s-a}\right)} = PS : \sqrt{\left(\frac{ab(s-b)}{s-a}\right)};$$

whence $PS = \frac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$.

Similarly, $OS = \frac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, and $NS = \frac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$.

The radius of the circumscribed circle of any triangle equals the product of the three sides divided by four times the area of the triangle.

Put the radius of the circumscribed circle of $\triangle ABC = R$, of $\triangle NSO = R_1$, of $\triangle PSO = R_2$, of $\triangle PSN = R_3$, and of $\triangle PON = R_4$.

Sides of $\triangle ABC$ are a, b, c; and area $= \sqrt{[s(s-a)(s-b)(s-c)]} = \triangle$.

$$\therefore R = \frac{abc}{4\triangle}.$$

Sides of $\triangle NSO$ are $\frac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$, $\frac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, $\frac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$; and area $= \frac{abc\sqrt{(s-a)}}{2\sqrt{[s(s-b)(s-c)]}}$.

$\therefore R_1=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PSO$ are $\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$; and

area $=\dfrac{abc\sqrt{(s-b)}}{2\sqrt{[s(s-a)(s-c)]}}$.

$\therefore R_2=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PSN$ are $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$, $\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$; and

area $=\dfrac{abc\sqrt{(s-c)}}{2\sqrt{[s(s-a)(s-b)]}}$.

$\therefore R_3=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PON$ are $\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$,

and area $=\dfrac{abc\sqrt{s}}{2\sqrt{[(s-a)(s-b)(s-c)]}}$.

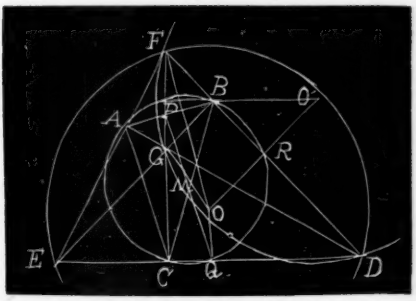
$\therefore R_4=\dfrac{abc}{2\triangle}=2R.$

Hence the general proposition : The centers of the escribed and inscribed circles of a triangle are the vertices of four different triangles the radii of whose circumscribed circles are equal.

Take $a=14$, $b=13$, $c=15$. Then $r=4$, $r_1=12$, $r_2=10\frac{1}{2}$, $r_3=14$, and $R=8\frac{1}{4}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be the given triangle, then if G is the incenter of ABC, E, F, D are its excenters. The circumcircle of ABC is the nine-points circle of EDF and G its orthocenter. Hence PQ is a diameter of the nine-points circle, since P is the mid-point of FG and Q the mid-point of ED.



$\therefore$ The mid-point of PQ is the center of the nine-points circle.

Let O be the circumcenter. Then the mid-point of GO is the center of the nine-points circle ; since perpendiculars to the mid-points of CQ and BR, chords of the nine-points circle, intersect at the mid-point of GO and also at the center of the nine-points circle.

$\therefore$ PQ and GO bisect one another at M. From the triangles MQO and GPM, $GM=OM$, $PM=QM$, $\angle OMQ=\angle PMG$. $\therefore$ $OQ=PG=FP$, and OQ is also parallel to FP. $\therefore$ $OF=PQ$.

$\therefore$ The radius of EFD=the diameter of ABC. Perpendicular to FG at the point P draw $PO'=QD$. Then since $FP=PG=OQ$, $O'F=O'G=OD$.

$\therefore$ $OF=O'F=OD=O'D=O'G$.

$\therefore$ radius of FGD=radius EFD=diameter ABC.

Similarly radius EGD and radius EGF=radius (each) EFD.

98. Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, N. J.

Construct a circle which shall pass through two given points and touch a given circle, (1) when the distance between the points is less than the diameter of the circle, and (2) when it is greater.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The construction for both cases is the same.

I Case. Let A, B be the given points, $CMDK$ the given circle. Through A, B draw the circle $ABEF$ intersecting the given circle in E, F. Draw AB, EF intersecting at G. Draw the tangents GK, GM. Draw QQ_1 perpendicular to AB at its mid-point. Through R, the center of $CMDK$, draw RM, RK intersecting QQ_1 in O_1, O. Then O, O_1 are the centers of two circles satisfying the conditions, and ABK, ABM are the circles.

II Case. Let C, D be the given points, $AHBL$ the given circle. Through C, D describe the circle $CDEF$ intersecting the given circle in E, F. Draw CD, EF intersecting at G. Draw the tangents GH, GL. Draw RR_1 perpendicular to CD at its mid-point. Through Q, the center of $AHBL$, draw QL, QH intersecting RR_1 in P, P_1. Then P, P_1 are the centers of two circles satisfying the conditions, and CDL, CDH are the circles.

In the above both points are without the given circle. This problem is thoroughly discussed on page 271, No. 8, Vol. I., THE AMERICAN MATHEMATICAL MONTHLY.

II. Solution by FREDERIC R. HONEY, Ph. B., New Haven, Conn.

The following description applies when the distance between the points is less, and when greater than the diameter of the circle.

Let a and b be the given points and A the circumference of the given circle.

Through a and b pass a circle the circumference of which intersects A at c and d. Draw ba and dc and produce these lines until they meet at e. Draw ef tangent to A. Through the point of tangency f and the given points a and b pass the required circle C. Since two tangents may be drawn there are, in each case, two solutions.

Analysis of the construction: $eb\times ea=ed\times ec=(ef)^2$.

[NOTE. For a demonstration of this same proposition with a diagram, see Vol. I., page 271. Professors Zerr and Honey each furnished neat diagrams with these demonstrations, but we believe the demonstrations sufficiently clear without them. ED. F.]

CALCULUS.

75. Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve the differential equation $\frac{d^2y}{dx^2}+n^2y=\frac{6}{x^2}y$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $y=ux^3$, the equation then becomes $\frac{d^2u}{dx^2}+\frac{6}{x}.\frac{du}{dx}+n^2u=0$.

Consider the equation $d^2v/dx^2+n^2v=0$.

Then $(dv/dx)^2+n^2v^2=A=n^2c^2$.

$\therefore \pm dv/\sqrt{(C^2-v^2)}=ndx$. $\therefore v=c\cos(nx+\alpha)$, or $c\sin(nx+\alpha)$.

Let $x^2=dz$, and change the independent variable from x to z.

$\therefore 2z(d^2v/dz^2)+dv/dz+n^2v=0$.

Differentiate this last three times with regard to z, and let $t=d^3v/dx^3$, then we get $2z.(d^2t/dz^2)+7.(dt/dz)+n^2t=0$.

Now re-change the independent variable from z to x.

$$\therefore d^2t/dx^2+\frac{6}{x}.\frac{dt}{dx}+n^2t=0.$$

$$\therefore u=t=\frac{d^3v}{dz^3}=\left(\frac{1}{x}.\frac{d}{dx}\right)^3[c\cos(nx+\alpha)].$$

$$\therefore y=x^3\left(\frac{1}{x}.\frac{d}{dx}\right)^3[c\cos(nx+\alpha)],$$

$$=cn^3\left[\left(1-\frac{3}{n^2x^3}\right)\sin(nx+\alpha)+\frac{3}{nx}\cos(nx+\alpha)\right],$$

$$=C\left[\left(1-\frac{3}{n^2x^2}\right)\sin(nx+\alpha)+\frac{3}{nx}\cos(nx+\alpha)\right].$$

If we had written $v=c\sin(nx+\alpha)$,

$$y=C\left[\frac{3}{nx}\sin(nx+\alpha)-\left(1-\frac{3}{n^2x^2}\right)\cos(nx+\alpha)\right].$$

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

This is a "case" of Riccati's Equation of the form

$$\frac{d^2u}{dx^2}-a^2x^{2q-2}u=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and of scientific interest as it appears in the theory of the figure of the earth. See Airy's *Figure of the Earth*, articles 64, 65; Col. Clark's *Geodesy*, page 80; Tisserrand's *Méc. Cel.*, *Tome* 2, Chapitre xiv *Equilibre d'Une Masse Fluide Hétérogène*, in the last of which the equation is

$$\frac{d^2y}{dx^2} = \left\{ \frac{a^2 \frac{d\rho}{da}}{\int_0^a \rho a^2 da} + \frac{6}{a^2} \right\} y \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It is due to this connection, that the proposed equation has received a variety of solutions.

What is here given is based on a transformation of (1) and a pair of solutions given in Johnson's *Differential Equations*, articles 213, 214.

(1) is transformed into

$$\frac{d^2u}{dx^2} - \frac{m-1}{x}\frac{du}{dx} - a^2u = 0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

two integrals of which are of the form

$$u_3 = e^{ax}\left(1 - \frac{m-1}{m-1}ax + \frac{(m-1)(m-3)}{(m-1)(m-2)}\frac{a^2x^2}{2!} - \ldots\ldots \right) \quad \ldots\ldots\ldots (4),$$

$$u_5 = e^{-ax}\left(1 + \frac{m-1}{m-1}ax + \frac{(m-1)(m-3)}{(m-1)(m-2)}\frac{a^2x^2}{2!} + \ldots\ldots \right) \quad \ldots\ldots\ldots (5),$$

and connected with u by the relation $u = Au_3 + Bu_5$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$.

Now assuming $u = x^p v \ldots\ldots\ldots (7)$, or $v = ux^{-p} \ldots\ldots\ldots (8)$, (3) becomes

$$\frac{d^2v}{dx^2} - a^2v = \frac{p(p+1)v}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Putting $a = n_1'(-1)$, $p = 2$, $m = 2p+1 = 5$, and $v = y$, $_1'(-1) = i$, (6) and (8), with (4) and (5) give

$$y = ux^{-2} = x^{-2}[Ae^{nix} + Be^{-nix} - \tfrac{1}{3}n^2x^2(Ae^{nix} + Be^{-nix}) + nx(Bie^{-nix} - Aie^{nix})] \quad \ldots (10),$$

which it is not difficult to reduce to the form,

$$y = Cx^{-2}[(3 - n^2x^2)\cos(nx + \alpha) + 3nx\sin(nx + \alpha)] \ldots\ldots\ldots\ldots (11).$$

See Forsyth's *Differential Equations*, Ed. 1885, Ex. 3, page 65; Ex. 1, page 175; Ex. 21, page 180; Ex. 26, page 181, and Ex. 5, pages 233, 4.

76. Proposed by E. B. ESCOTT, Cambridge, Mass.

Solve the partial differential equation, $q^2r + 4pqs + p^2t + p^2q^2(rt - s^2) = a^2$. [*Forsyth's Differential Equations*, page 376.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The equation in λ is, $\lambda^2p^2q^2(1+a^2) + 4\lambda p^3q^3 + p^4q^4 = 0$.

$\therefore \lambda^2(1+a^2) + 4\lambda pq + p^2q^2 = 0$.

$\therefore \lambda = -\{pq/(1+a^2)\}[1 \mp _1'(3 - a^2)] = m_1pq$ or m_2pq.

The first system of integrals is $p^2q^2dy + m_1p^3qdx + m_1p^3q^3dp = 0$.

$p^2q^2dx + m_2pq^3dy + m_2p^3q^3dq = 0$, or $qdy + m_1pdx + m_1pq^2dp = 0 \ldots\ldots\ldots (1)$;

$pdx+m_2qdy+m_2p^2qdq=0$ ....(2).

Similarly for second system $pdx+m_1qdy+m_1p^2qdq=0$.....(3);

$qdy+m_2pdx+m_2pq^2dp=0$ ....(4).

(1)+(3) gives $(pdx+qdy)(1+m_1)+m_1(pq^2dp+p^2qdq)=0$, but $dz=pdx+qdy$.

$\therefore (1+m_1)dz+m_1pq^2dp+p^2qdq)=0$.

$\therefore 2z(1+m_1)+m_1p^2q^2=2a$ .....(5).

Similarly (2)+(3) gives $2z(1+m_2)+m_2p^2q^2=2b$.....(6).

Eliminating p^2q^2 between (5) and (6) we get $z(m_1-m_2)=bm_1-am_2$.

77. Proposed by T. E. COLE, Columbus, Ohio.

Derive the equation of a point in a pedal of a bicycle as the wheel rolls along on a plane.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and A. E. BREECE, A. B., Professor of Mathematics, Portland University, Portland, Oregon.

Let CO be the radius of a circle whose circumference is equal in length to the distance C moves in one revolution of the pedal. P a point in the pedal, A, A' points where CP is perpendicular to the fixed line AA'.

Let $CO=a$, $CP=d$, $\angle OCP=b$, then we have

$x=AN=AO-ON=a\theta-d\sin b$; $y=PN=CO+PM=a-d\cos\theta$.

$\therefore x=a\cos^{-1}[(a-y)/d]-\sqrt{}[d^2-(a-y)^2]$.

If $a=d$, the curve is a right cycloid.

If $a<d$, the curve is a prolate cycloid.

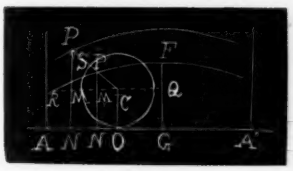
If $a>d$, the curve is a curtate cycloid.

II. Solution by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

Consider the curve traced by the pedal in one turn. Take origin at the lowest point of the pedal's path. Let axis of y be vertical and axis of x the straight line joining any two consecutive lowest points of pedal's path. Take P any point in the curve and draw its ordinate PN. Let F be the middle point of curve and FG its ordinate. From Q, the middle point of FG, draw QR parallel to the x-axis. Take $PC=FQ$ and draw CO parallel to PN.

Now let a be length of pedal arm, r the radius of bicycle wheel, n the number of turns the wheel makes to one turn of the pedal, and θ the angle PCO through which the pedal arm has turned on reaching the point P in the curve.

Then from the figure, we have,



$$x=AN=AG-(GO+MC)=\pi rn-\left[\left(\pi rn-\frac{\pi rn\theta}{180}\right)+a\sin\theta\right]=\frac{\pi rn}{180}\theta-a\sin\theta.$$

$$y=PN=MN+MP=a-a\cos\theta=a\text{vers}\theta.$$

Whence the equation of curve is, $x=\frac{\pi rn}{180}\text{vers}^{-1}\frac{y}{a}-\sqrt{}(2ay-y^2)$.

MECHANICS.

66. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The distance, parallel to the axis, from the mid-point of a chord to the arc of a parabola is constant. Show that the center of gravity of all segments formed by the chord is an equal parabola.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

Taking the tangent parallel to the chord as y-axis, and the corresponding diameter as x-axis, denoting the inclination of the axes by β, and the latus rectum by $2p$, the equation of the parabola is

$$y^2 = \frac{2p}{\sin^2\beta}x.$$

Since all chords parallel to the base of the segment are bisected by the x-axis, the center of gravity is on this axis.

$\bar{x}$ being the abscissa of the center of gravity and a the distance from the origin to the mid-point of the chord, we have, by taking moments about the tangent,

$$\bar{x}\sin\beta = \frac{\int_0^a y\sin\beta\,dx.x\sin\beta}{\int_0^a y\sin\beta\,dx}, \text{ or } \bar{x} = \frac{\int_0^a \sqrt{(2p)}x^{\frac{3}{2}}\,dx}{\int_0^a \sqrt{(2p)}x^{\frac{1}{2}}\,dx} = \tfrac{3}{5}a.$$

Since a is constant for all positions of the chord, if the given parabola be moved to the right a distance equal to $\frac{3}{5}a$, it will coincide with the locus of the centers of gravity. Hence this locus is an equal parabola.

67. Proposed by B. F. FINKEL, A. M.,M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A conical stick of timber, length a, radius of base r, and density δ, is depressed, apex downward, in a liquid, density δ', so that the base is just level with the liquid. If left free to rise, required the greatest altitude to which it will ascend.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Let x=the part of the axis above the liquid at any time t from the beginning of the motion; then the volume of the cone being $\frac{1}{3}\pi r^2 a$, the part under the liquid is $\frac{1}{3}\pi r^2 a[(a-x)/a]^3$, and if ρ be the density of the cone, the force acting on the cone is proportional to the members of the equation

$$\tfrac{1}{3}\pi r^2 a\,\rho\frac{vdv}{dx} = g[\tfrac{1}{3}\pi r^2 a\delta'\left(\frac{a-x}{a}\right)^3 - \tfrac{1}{3}\pi r^2 a\rho] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

v being the velocity.

Or (1) is $a^3\rho\frac{vdv}{dx}=g[\delta'(a-x)^3-\rho a^3]$ .....(2).

Integrating (2), $\frac{1}{2}a^3\rho v^2=g[-\frac{1}{4}\delta'(a-x)^4-\rho a^3x]+C$.....(3).

When $x=0$, $v=0$, $\therefore C=\frac{1}{4}(\delta' a^4 g)$ and (3) becomes

$\frac{1}{2}(a^3\rho)v^2=g\{(\frac{1}{4}\delta')[a^4-(a-x)^4]-\rho a^3x\}$..... .....(4).

At the greatest height, $v=0$, and then

$(\frac{1}{4}\delta')[a^4-(a-x)^4]-\rho a^3x=0$.....(5),

a biquadratic for x. One value plainly being 0, there remains a cubic to be solved.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $DK=x$, $DL=y$, $\therefore x+y=a$. When the cone is under the liquid its acceleration is $g(\delta'-\delta)/\delta$.

The equation of motion is $d^2x/dt^2=\beta y^3-g$.

When $y=a$, $d^2x/dt^2=g(\delta'-\delta)/\delta$.

$\therefore \beta=\delta' g/a^3\delta$.

$\therefore d^2x/dt^2=(g/a^3\delta)(\delta' y^3-\delta a^3)=(g/a^3\delta)[\delta'(a-x)^3-\delta a^3]$.

$\therefore (dx/dt)^2=-(g/2a^3\delta)[(a-x)^4\delta'+4a^3x\delta]+A$.

When $y=a$, $dx/dt=0$, $x=0$.

$\therefore A=ag\delta'/2\delta$.

$\therefore (dx/dt)^2=v^2=(ag\delta'/2\delta)-(g/2a^3\delta)[\delta'(a-x)^4+4\delta a^3x]$.

When $v=0$, $\delta'(a-x)^4+4\delta a^3x=a^4\delta'$, or

$$x^3-4ax^2+6a^2x=4a^3(\delta'-\delta)/\delta'.$$

This cubic determines x. When $\delta<\frac{1}{4}\delta'$, the cone leaps out of the liquid. At this moment $x=a$, $\therefore v^2=ag(\delta'-4\delta)/2\delta$.

Then $h=v^2/2g=a(\delta'-4\delta)/4\delta$=height of vertex above water.

$\therefore$ Total height of base above liquid$=a+h$.

68. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the horizontal and vertical components of the moon's "disturbing force" for any point on the earth's surface making an angle φ with the line joining the center of the earth to the center of the moon.

Solution by the PROPOSER.

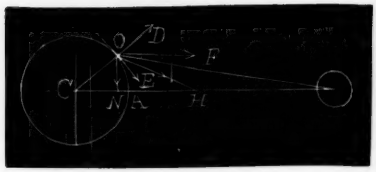
Let CM, the distance from the earth to the moon$=d$; CA, the radius of the earth$=a$; O the point on the earth; $\angle OCA=\varphi$; m=moon's mass; f="moon's disturbing force" in direction MN; f'=disturbing force in direction ON.

Then $f'=(m/OM^2)\times(ON/OM)=m(ON/OM^3)$.

$ON=a\sin\varphi$, and we can take $OM=d$ without any appreciable error.

$$\therefore f' = \frac{ma\sin\varphi}{d^3} = Q\sin\varphi,$$

where $Q = ma/d^3$.



$$f = m\left(\frac{1}{NM^2} - \frac{1}{CM^2}\right) = m\left(\frac{CM^2 - NM^2}{CM^2 . NM^2}\right)$$

$$= \frac{mCN(CM+NM)}{CM^2 . NM^2} = \frac{m . CN(2d - CN)}{d^2(d-CN)^2}$$

$$= \frac{2m . CN}{d^3} \cdot \frac{(1 - CN/2d)}{(1 - CN/d)^2}.$$

Since CN/d is small we may neglect it

$$\therefore f = \frac{2m . CN}{d^3} = \frac{2ma\cos\varphi}{d^3} = 2\,Q\cos\varphi.$$

Now let f = horizontal, f' = vertical components.

$$\therefore f = f\sin\varphi + f'\cos\varphi = 3Q\sin\varphi\cos\varphi = \tfrac{3}{2}Q\sin 2\varphi.$$

$$f' = f\cos\varphi - f'\sin\varphi = Q(2\cos^2\varphi - \sin^2\phi) = Q(3\cos^2\varphi - 1).$$

DIOPHANTINE ANALYSIS.

68. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find a *general* value for p in the expression $4p+1$ = the sum of two squares.

I. Solution by J. H. DRUMMOND, LL. D., Portland, Me., and G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Since $4p+1$ is odd one of the squares must be even and the other odd. Take $2q+1$ as one of the numbers and $2s$ for the other, and we have $4q^2+4q+1+4s^2 = 4p+1$. Hence $p = q^2+q+s^2$ in which q may be zero or any number; and s any number.

II. Solution by the PROPOSER.

From a table in which I have all the odd numbers, up to 12013, that are equivalent to the sum of two squares, I find by inspection that all the values for $4p+1$ can be obtained by making $p = \frac{n(n+1)}{2} + \frac{a(a+1)}{2}$, in which $n > a$.

Whence $4\left[\frac{n(n+1)}{2} + \frac{a(a+1)}{2}\right] + 1 = (n+a+1)^2 + (n-a)^2$. (I).

When $a=0$, we have $4\left[\frac{n(n+1)}{2}\right] + 1 = (n+1)^2 + n^2$; and we obtain the series of values 5, 13, 25, 41, 61, 85, etc.

When $a=1$, we have $4\left[\frac{n(n+1)}{2} + 1\right] + 1 = (n+2)^2 + (n-1)^2$; and we obtain the series of values 17, 29, 45, 65, 89, 117, etc.

When $a=2$, we have $4\left[\frac{n(n+1)}{2}+3\right]+1=(n+3)^2+(n-2)^2$; and we obtain the series of values 37, 53, 73, 97, 125, 157, etc.

We observe that the development of $4p+1$ occurs in *series*; the *number of terms in each series* as well as the *number of series* being infinite.

We also notice that the *number of series*$=a+1$. Now put $r=n-a$. We then readily deduce that the rth *term of any series*

$$=4\left[\frac{(r+a)(r+a+1)}{2}+\frac{a(a+1)}{2}\right]+1=(r+2a+1)^2+r^2. \quad \text{(II)}.$$

For the numerical development of $4p+1$, Formula II is better adapted than Formula I, as the values of r and a are independent of each other. But we can still improve a little in this direction. Put $N=a+1=$*the number of the series.*

Then (A), the rth term of the Nth series, and also, (B), the Nth term of the rth series

$$=4\left[\frac{(N+r)(N+r-1)}{2}+\frac{N(N-1)}{2}\right]+1=4\left[N(N+r-1)+\frac{r(r-1)}{2}\right]+1$$

$$=(2N+r-1)^2+r^2. \quad \text{(III)}.$$

From this we observe that there are *two forms of series* embodied in one formula, each form, however containing all the values of $4p+1$.

The one form, of which we have already treated and in which $r=$the consecutive integers for each value of N, or for each series, we shall designate as "Series A."

The other form, in which $N=$the consecutive integers for each value of r, or for each series, we shall term "Series B."

The *first series* of "Series B" consists of the *first terms* of the consecutive series of "Series A"; as 5, 17, 37, 65, 101, 145, etc.

The *second series* of "Series B" consists of the second terms of the consecutive series of "Series A"; as 13, 29, 53, 85, 125, 173, etc.; and so on for the respective series.

Two other values of p, in Formula III, are

$$\frac{(2N+r)(r-1)}{2}+N^2 \text{ and } (N-1)(N+r)+\frac{r(r+1)}{2},$$

obtained by different arrangement of terms and factoring. The two values are significant; as $4\left[\frac{(2N+r)(r-1)}{2}\right]$ is the difference between the rth and first terms in "Series A," and $4(N-1)(N+r)$ is the difference between the Nth and first terms of "Series B."

We have also a general formula for finding *consecutively* the terms of a series:—

In "Series A", rth term $+4(N+r)=r+1$th term.

In "Series B", Nth term $+4(2N+r)=N+1$th term.

The following mathematical diversions may be interesting as bearing upon this problem.

Knowing the first series in "Series A", we can find the other series consecutively by means of the following rule: *Add to each term of the last found series, omitting the first term, the number of the series times* 4, or $4N$; as

First series.	5, 13, 25, 41, 61, 85, 113, etc.
Add 1×4.....	4, 4, 4, 4, 4, 4,
Second series.....	17, 29, 45, 65, 89, 117, etc.
Add 2×4.....	8, 8, 8, 8, 8,
Third series.....	37, 53, 73, 97, 125, etc., etc.

The number of the series $=N=\frac{1}{4}$(1st term -1); or $4N^2+1=$1st term.

In "Series B", $4\left[\frac{r(r+1)}{2}\right]+1=$1st term.

The following is a most interesting deduction from Formula III.

(1), $(2N+r-1)^2+r^2=(2N-1)(2N+2r-1)+2r^2=2r(2N+r-1)+(2N-1)^2$,

(2), $[(2N-1)(2N+2r-1)]^2+[2r(2N+r-1)]^2=[(2N+r-1)^2+r^2]^2$, or = the *square of* $4p+1$=the sum of two squares.

The application of these deductions gives rise to the annexed table, which I have constructed and extended to over three thousand numbers each equal to $4p+1$=the sum of two squares. It contains every number of the kind up to $12013=77^2+78^2$. By means of the table we can readily find—

(1). The two numbers the sum of whose squares=the given numerical value of $4p+1$.

(2). The two numbers the sum of whose squares=the square of the given numerical value of $4p+1$.

The *first column* consists of the consecutive values of r, and is called the "r column" of consecutive integers.

The other columns are the consecutive series of "Series A."

The *first row* is the "N row" of consecutive integers.

The *second row* is the "$2N-1$ row" of consecutive odd numbers.

The other rows are the consecutive series of "Series B."

The number of the column of values of $4p+1$ is indicated at its top by the value of N; and the number of the row of values is shown at its left by the value of r.

In using the table, all mention of values of r, N, and $2N-1$, refer to the respective values in the *same row* and the *same column* in which is found the given value of $4p+1$.

To find the two numbers the sum of whose squares$=4p+1$; one of the numbers$=r+2N-1$ and the other number$=r$. Take 97, at the intersection

of *column* 3 with *row* 4. Then $4+5=9$=one number, and 4=the other number; $97=9^2+4^2$.

To find the two numbers the sum of whose squares=the square of $4p+1$; one of the numbers=$4p+1-2r^2$, and the other number=$4p+1-(2N-1)^2$.

Take 97. Then $97-2\times4^2=65$=one of the numbers, and $97-5^2=72$; $97^2=65^2+72^2$.

TABLE OF VALUES OF $4p+1$=THE SUM OF TWO SQUARES.

	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	3	5	7	9	11	13	15	17	19	21	23	25
1	5	17	37	65	101	145	197	257	325	401	485	577	677
2	13	29	53	85	125	173	229	293	365	445	533	629	733
3	25	45	73	109	153	205	265	333	409	493	585	685	793
4	41	65	97	137	185	241	305	377	457	545	641	745	857
5	61	89	125	169	221	281	349	425	509	601	701	809	925
6	85	117	157	205	261	325	397	477	565	661	765	877	997
7	113	149	193	245	305	373	749	533	625	725	833	949	1073
8	145	185	233	289	353	425	505	593	689	793	905	1025	1153
9	181	225	277	337	405	481	565	657	757	865	981	1105	1237
10	221	269	325	389	461	541	629	725	829	941	1061	1189	1325
11	265	317	377	445	521	605	697	797	905	1021	1145	1277	1417
12	313	369	433	505	585	673	769	873	985	1105	1233	1369	1513
13	365	425	493	569	653	745	845	953	1069	1193	1325	1465	1613
14	421	485	557	637	725	821	925	1037	1157	1285	1421	1565	1717
15	481	549	625	709	801	901	1009	1125	1249	1381	1521	1669	1825
16	545	617	697	785	881	985	1097	1217	1345	1481	1625	1777	1937
17	613	689	773	865	965	1073	1189	1313	1445	1585	1733	1889	2053
18	685	765	853	949	1053	1165	1285	1413	1549	1693	1845	2005	2173
19	761	845	937	1037	1145	1261	1385	1517	1657	1805	1961	2125	2297
20	841	929	1025	1129	1241	1361	1489	1625	1769	1921	2081	2249	2425
21	925	1017	1117	1225	1341	1465	1597	1737	1885	2041	2205	2377	2557
22	1013	1109	1213	1325	1445	1573	1709	1853	2005	2165	2333	2509	2693
23	1105	1203	1313	1429	1553	1685	1825	1973	2129	2293	2465	2645	2833
24	1201	1305	1417	1537	1665	1801	1945	2097	2257	2425	2601	2785	2977
25	1301	1409	1525	1649	1781	1927	2069	2225	2389	2561	2741	2929	3125
26	1405	1517	1637	1765	1904	2045	2197	2357	2525	2701	2885	3077	3277
27	1513	1629	1753	1885	2025	2173	2329	2493	2665	2845	3033	3229	3433
28	1625	1745	1873	2009	2153	2305	2465	2633	2809	2993	3185	3385	3593

MISCELLANEOUS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are poured in at the two cusps, the quantity of each being such as would fill a length of the tube equal to its axis a. If the fluids do not mix, find the distance x_1, x_2 of the upper levels of the fluids from the vertex measured along the cycloidal arc. [From *Procter's Geometry of the Cycloid.*]

Solution by the PROPOSER.

Let P be the atmospheric pressure. Then the pressure at the first surface of separation is $P+g\sigma_1[x_1^2-(x_1-a)^2]/8a$.

Similarly at the second or free surface, $P+g\{\sigma_1[x_1^2-(x_1-a)^2]+\sigma_2[(x_1-a)^2-(x_1-2a)^2]\}/8a$.

$\therefore P=P+g\{\sigma_1[x_1^2-(x_1-a)^2]+\sigma_2[(x_1-a)^2-(x_1-2a)^2]\}/8a$.

$\therefore 2x_1(\sigma_1+\sigma_2)=a(\sigma_1+3\sigma_2)$.

Similarly $P=P+g\{\sigma_2[x_2^2-(x_2-a)^2]+\sigma_1[(x_2-a)^2-(x_2-2a)^2]\}/8a$.

$2x_2(\sigma_1+\sigma_2)=a(3\sigma_1+\sigma_2)$.

63. Proposed by F. M. SHIELDS, County Surveyor, Coopwood, Miss.

Of three chronometers, A, B, and C, A keeps true time; B gains 5 minutes and $15\frac{3}{4}$ seconds a day by true time, and C loses 7 minutes and $15\frac{3}{4}$ seconds a day by true time. The hands of all three watches are set at 12 noon on a certain day. What is the time by the true watch, A, on the *fifth* day after the time when the hands of the fast watch, B, point to 12, and what is the time by the true watch, A, on the *tenth* day after the time when the hands of the slow watch, C, point to 12?

I. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

B goes $1445\frac{21}{80}$ minutes while A goes 1440; then $1445\frac{21}{80} : 1440 :: 5 : 4\text{d. } 23\text{h. } 33\frac{90567}{115621}\text{m.}$, and the answer is $33\frac{90567}{115621}$ minutes past eleven.

C goes $1432\frac{26}{35}$ while A goes 1440; hence $1432\frac{26}{35} : 1440 :: 10 : 10\text{d. } 1\text{h. } 12\frac{23544}{25073}\text{m.}$, and the answer is $12\frac{23544}{25073}$ minutes past one.

II. Solution by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

While the watch A goes 24 hours, or 1440 minutes, the watch B goes 1445 minutes, $15\frac{3}{4}$ seconds, or $1445\frac{21}{80}$ minutes. To find how much the watch A is behind B at the end of 24 hours *by* B, we find 1 minute of B equal to $1445\div 1445\frac{21}{80}$ of A's, and 1440 minutes of B's $=1440\div 1445\frac{21}{80}=1434\frac{87486}{115621}$, *i. e.* when the hands of the watch B point to 12 at the end of the first 24 hours *by* B, the watch A lacks $5\frac{28135}{115621}$ minutes. On the *fifth* day after time when the hands of the watch B point to 12, the watch A lacks $5\frac{28135}{115621}\times 5=26\frac{25054}{115621}$ minutes to 12.

At the end of the first 24 hours *by* C the watch A is $7\frac{73693}{25073}$ past 12. On the tenth day after time when the hands of the slow watch C point to 12, the time by the true watch A is $7\frac{73693}{25073}\times 10=72\frac{23544}{25073}$ minutes, or 1 hour, $12\frac{23544}{25073}$ minutes past 12.

III. Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

24 hours, 5 minutes and $15\frac{3}{4}$ seconds on chronometer B equals 24 hours on chronometer A. Hence 5 days on B equals 4 days, 23 hours, 33 minutes and 46.9 seconds on A. Therefore the time by the true watch A is 11 hours, 33 minutes, and 46.9 seconds A. M. on the fifth day.

$1432\frac{26}{35}$ minutes on chronometer C equals 1440 minutes on chronometer A.

Hence ten days on chronometer C equals 10 days, 1 hour, 12 minutes and 55 seconds on chronometer A. Therefore the time by the true watch A is 12 minutes, 55 seconds past 1 P. M. on the tenth day.

Also solved by *CHARLES C. CROSS*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

90. Proposed by J. MARCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Fully solve $x^2+y^2+m(x+y)=m^2$ ..... (I); $x^2+y^2+xy=m^2$ ..... (II); give process, roots, and corollary. [Solved *in part*, *Cirode's Algebra*, *page* 202.]

91. Proposed by NELSON S. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.

Solve the following without making use of the determinant notation and prove that the results obtained are the roots.

$$10x-2y+4z=5,$$
$$3x+5y-3z=7,$$
$$x+3y-2z=2.$$

92. Proposed by W. F. BRADBURY, A. M., Head Master Latin School, Cambridge, Mass.

Find the sum to n terms of $1+3^3+5^3+$..... [From *Charles Smith's Elementary Algebra*, page 403.]

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

GEOMETRY.

105. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Sind A, B, C, D vier harmonische Punkte und beschreibt man über dem Durchmesser AC einen Kreis, von welchem S ein beliebiger Punkt ist, so wird derjenige Kreisbogen, welcher innerhalb des Winkels BSD liegt, entweder von A order von C Strecke PB halbirt. [*Reye's Geometrie der Lage*, page 191.]

106. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Upon the sides of any triangle ABC let the equilateral triangles ABD, BCE, and CAF be described, and let their exterior sides produced intersect, BE and AF in K, DB and FC in L, and DA and EC in M. Prove DK, EL, FM, parallel.

107. Proposed by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama.

Construct a triangle, given base, vertical angle and radius of inscribed circle.

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

CALCULUS.

81. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, Carthage, Texas.

Solve: $y^2(d^2y/dx^2)+a(dy/dx)^2=bx$.

82. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

A pole 60 feet high stands vertically in a river 20 feet deep. How many feet above the surface of the water must it break so that the top bending down would touch the bottom and the distance on the surface of water between the points where the parts of the pole enter the water would be maximum?

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

MECHANICS.

75. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A particle, P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation to the form of tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

76. Proposed by J. F. LAWRENCE, Classical Sophomore, Drury College, Springfield, Mo.

An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the inclined plane, x the part of the plane rolled over by the sphere, prove that $(M+m)x'=mx\cos\alpha$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$, where α is the inclination of the plane. [From *Routh's Elementary Rigid Dynamics*, page 126.]

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

DIOPHANTINE ANALYSIS.

74. Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve $x^2+y^2=a^2$,
$z^2+w^2=b^2$,
$y^2+w^2=d^2$, *i. e.* obtain values of x, y, z, w, which will make the second members perfect squares.

75. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

(1). In how many ways can the consecutive integers 1 to 16 be arranged as a Magic Square?

(2). Arrange the consecutive integers 1 to n^2 as a Magic Square, where n is odd. Apply when $n=9$.

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

AVERAGE AND PROBABILITY.

71. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average volume removed by boring an inch auger-hole through a cube whose edge is e, the auger to pass through two opposite faces of the cube.

72. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A rod is broken at random into four pieces; find the chance that no one of the pieces is greater than the sum of the other three. [From *C. Smith's Treatise on Algebra*, page 528.]

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

MISCELLANEOUS.

68. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Find the locus of the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ so that the plane of contact will constantly touch $x^2+y^2+z^2=r^2$.

69. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O., Sebastopol, Cal.

Find the locus of a point equi-distant from the circumferences of two fixed circles.

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

BOOKS.

Advanced Mechanics. Vol. I. *Dynamics.* The Organized Science Series. By William Briggs, M. A., F. E. S., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. 327 pages. Price, $1.00. 1898. London: W. B. Clive. New York: Hinds & Noble.

The notion of relative velocities is introduced at an early stage in the discussion of the laws of composition and resolution. The chapters on motion down inclined planes, chords of quickest descent, the parabolic path of a projectile, circular and harmonic motion, small oscillations of a pendulum, impact of elastic spheres, and the elements of rigid dynamics are highly satisfactory. There are numerous exercises. The plan of the book is good, and in the execution there is much to commend.

We have also received *Elementary Mechanics*, by the same authors, in the University Tutorial Series, which gives special prominence to fundamental principles, and is an especially good book of its grade. J. M. C.

Coördinate Geometry. The Right Line and Circle. By Briggs and Bryan. University Tutorial Series. Third Edition. 220 pages. Price, 80 cents.

In this elementary treatise the editors have tried "to realize the position of the average learner, and have constantly borne in mind the needs of the private student." The result is altogether satisfactory. J. M. C.

Mensuration and Spherical Geometry. By William Briggs, M. A., F. C. S., F. R. A. S., and T. W. Edmondson, B. A. Second Edition. $1.00. London: W. B. Clive. New York: Hinds & Noble.

The subject is not presented in the form of a mere collection of rules and examples, but in every case a proof of the formula to be used has been given. There are no material alterations in the revision, but some errors have been eliminated and some inaccuracies corrected. The work as now presented is well arranged and fairly comprehensive in treatment. It will be found a serviceable text. J. M. C.

Differential and Integral Calculus. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. Revised Edition. 1898. 8vo., cloth. xiii+269 pages. Price, $2.15. Boston: Ginn & Company.

In this revision of Professor Taylor's popular text the attempt has been made to present in their unity the three common methods in the Calculus. In his preface, the author says that "the concept of Rates is essential to a statement of the problems of the Calculus; the principles of Limits make possible general solutions of these problems, and the laws of Infinitesimals greatly abridge these solutions." The Method of Rates is so generalized and simplified that it does not involve "the foreign element of time," and affords simple and brief proofs of first principles. By proving $lt(\triangle y/\triangle x)=dy/dx$, the problem of rates is reducedto one of limits or infinitesimals. The Theory of Infinitesimals is viewed simply as that part of the theory of limits which deals with *variables having zero as their common limit.* Emphasis is also laid upon the distinction between infinitesimals and zero and that between infinites and $a/0$. A chapter on differential equations is an important addition to the text. The table of integrals is useful. Throughout the work are found numerous practical problems from mechanics and other branches of applied mathematics. One of the most distinctive features of this text (though it may be a source of objection to some) is the facility with which it may be used by teachers who may prefer either one of the three methods commonly used in the Calculus, as those who wish to pursue the study by the method of Limits or Infinitesimals alone can omit the few demonstrations which involve rates and take in their stead those by limits or infinitesimals. We bespeak for the book a hearty welcome from teachers. J. M. C.

Differential and Integral Calculus, for Technical Schools and Colleges. By P. A. Lambert, M. A., Assistant Professor of Mathematics, Lehigh University. 8vo. Cloth. x+246 pages. Price, $1.50. New York: The Macmillan Co.

This work is designed for students who have a working knowledge of Elementary Geometry, Algebra, Trigonometry and Analytical Geometry. The author states that the object of the book is: (1) By a logical presentation of principles to inspire confidence in the methods of the infinitesimal analysis; (2) By numerous problems to aid in acquiring facility in applying these methods; (3) By applications in Physics and Engineering, and other branches of Mathematics, to show the practical value of the Calculus.

Proposing problems belonging to Engineering is a good idea, and it would be well in making up lists of exercises to go in future works on the Calculus, to insert more problems in Economics and Engineering. B. F. F.

Text-Book of Algebra with Exercises for Secondary Schools and Colleges. By George Egbert Fisher, M. A., Ph. D., and Isaac J. Schwatt, Ph. D., Assistant Professors of Mathematics in the University of Pennsylvania. Part I. 8vo. Cloth. Price, $1.25. xiii+683 pages. Published by the authors.

This work begins with the very elements of algebra, establishing by rigorous demonstrations, and making clear by numerous illustrations, the fundamental Laws and Theorems, through the Fundamental Operations, Fractions, Equations of one, two, etc., unknown

quantities, Evolution, Inequalites, Quadratic Equations, Proportion and Variation, Progressions, and ends with a discussion of the Binominal Theorem for Positive Integral Exponents. Some of the strong points in favor of this work are: The solutions of equation based upon equivalent equations and equivalent systems of equations; the discussion of general problems and the interpretation of positive, negative, zero, indeterminate, and infinite solutions of problems; the logical discussion of irrational numbers; and the full treatment of inequalities. An unusually large number of problems and exercises have been added, in order that the teacher from year to year, may have variety with different classes. The work is one of unusual merit and is worthy the patronage of teachers desiring to put a good book into the hands of their pupils. The print is large and clear, the printed page is not overcrowded, and the binding is good. B. F. F.

Introduction to the Theory of Analytic Functions. By J. Harkness, M. A. (Cambridge), Professor of Mathematics, Bryn Mawr College, Pa., and F. Morley, Sc. D. (Cambridge), Professor of Mathematics, Haverford College, Pa. Large 8vo. Cloth. xv+336 pages. Price, $3.00. New York: The Macmillan Co.

This work, as its name indicates, is an introduction to the Theory of Functions. Until the translation of Durege's Elements of the Theory of Functions by Drs. Fisher and Schwatt, in 1896, there was no work in English giving a consecutive and elementary account of the fundamental concepts and processes employed in the Theory of Functions. The translation referred to above has been very helpful in aiding the student in making some headway in the larger works in English, viz., Forsyth's Treatise, and Harkness and Morley's Treatise. To still further aid in introducing the student to the advanced works, the work before us is intended. The work begins with a treatment of the Ordinal Number System. This discussion while very elementary, is of the highest importance, and might well be incorporated in some works on arithmetic, for it is true that few arithmetics at present bear evidence of the great progress and improvement made in mathematics. Chapter II treats of the Geometric Representation of Complex Number; Chapter III, of the Bilinear Transformation; Chapter IV, of Geometric Theory of the Logarithm and the Exponential; Chapter V, of the Bilinear Transformation of the Plane into Itself; Chapter VI, of Limits and Continuity; Chapter VII, of Rational Algebraic Functions; Chapter VIII, of Convergence of Infinite Series; Chapter IX, of Uniform Convergence of Real Series; Chapter X, of Power Series; Chapter XI, of Operations with Power Series; Chapter XII, of Continuation of Power Series; Chapter XIII, of Analytic Theory of the Exponential and Logarithm; Chapter XIV, of Singular Points of Analytic Functions; Chapter XV, of Weirstrass's Factor-Theorem; Chapter XVI, of Integration; Chapter XVII, of Laurant's Theorem and the θ-Functions; Chapter XVIII, of Functions Arising from a Network; Chapter XIX, of Elliptic Functions; Chapter XX, of Simple Algebraic Functions on Reimann Surfaces; Chapter XXI, of Algebraic Functions; Chapter XXII, of Cauchy's Theory and the Potential. The work forms an excellent introduction to the authors' Treatise. B. F. F.

An Elementary Course in Analytical Geometry. By J. H. Tanner, Assistant Professor of Mathematics in Cornell University, and Joseph Allen, formerly Instructor in Mathematics in Cornell University, Tutor in the College of the City of New York. 8vo. Cloth. xx+390 pages. Price, $2.00. New York, Cincinnati, and Chicago: The American Book Co.

This is the first of the Cornell Mathematical Series. In addition to the usual discussion of the Conic Section. there is 22 pages devoted to the discussion of Higher Plane Curves; 50 pages to Solid Analytical Geometry, and an Appendix of 10 pages in which is a short historical account of the invention of Analytical Geometry, discussions of the limiting cases of series, and the demonstration of the proposition due to Hamilton, Quételet, and others. The work is interspersed with brief historical notes adding much interest to the study of the subject. B. F. F.

Errata.

Page 106, line 20, for "2(3)" read $2(3)^2$; line 21, for "$2(a^4+b^4+c^2)$" read $2(a^4+b^4+c^4)$; line 29, last term of denominator, for "$3(a^2+b^2+c^2)^2$" read $3(a^2+b^2-c^2)^2$.

Page 107, bracket together lines 9, 10 and 11, and number (15) ; line 22, for "$a^4c^2+2a^2c^4-c^6$" read $a^4c^2-2a^2c^4+c^6$.

Page 118, line 2, for "$\cos n\alpha$)" read $\cos^n\alpha$).

Page 139, line 3 from bottom, for "P_2a^{n-4}" read $P_2a_2^{n-4}$.

Page 140, line 2, for "P_{n-2}" read $\pm P_{n-2}$; line 6, in second term for "$a^ra_1^{n-2}$" read $a_2^ra_1^{n-2}$.

Page 151, line 28, where "18" occurs read 78.

Page 155, line 6 from bottom, in numerator of (A), for "$(n-2)$" read $(n-1)$.

Page 157, line 8, for "$ED : DB$" read $EB : OB$.

Page 159, problems 62 and 63 should be 64 and 65.

Page 173, line 30, for "$-2[S$" read $\pm 2[8$.

Page 174, line 3, for "946268" read 04268 ; line 5, for "8" read -8 ; line 6, for "0.512372" read ∓ 15.487627 ; line 9, for "-0.064568," etc., read 0.064568, etc.; line 10, for "300343" read 800343 ; line 18, for "$\frac{1}{4}f_1$ $[1(a^2-b)]$" read $\frac{1}{4}f_1$ $[1/(a^2-b)]$; lines 19 and 20, for "$(\frac{1}{4}+)$" read $(1/4+)$.

Page 175, line 2, of problem 92, for "$AB.BC : DC.AD=BD : AC$" read $AB.BC+AD.CD : AB.AD+BC.CD :: BD : AC$.

Page 177, line 5, for "EBC" read FBC ; line 12, insert sign $=$ before $\frac{1}{4}(2880-y^2)$; line 21, "353.3604" read 353.8604 ; supply F in figure.

Page 180, line 13, insert comma after $(2n^2+4l+1)^2$; line 29, for "$\frac{1}{3}$ $[(q^2+4q^2+4q+1)$" read $\frac{1}{3}[(q^3+4q^2+4q+1)$.

Page 181, line 5, for "$-3m^3a^2y$" read $-3m^2u^2y$; line 17, problem should be 64.

Page 182, line 2 from bottom, read $(y/r)^{\frac{3}{4}}=a^2/(1+a^2)$.

Page 183, line 2, for (3) read $a/(1-a^2)=\{(y/r)^{\frac{1}{3}}/[1-(y/r)^{\frac{3}{4}}]\}/[1-2(y/r)^{\frac{3}{4}}]$; line 19, for "$\alpha\chi$" read α/χ ; line 3 from bottom, for "$y^3=\frac{1}{2}y+.094119=0$" read $y^3-\frac{1}{2}y+.094118=0$.

Page 186, line 4, insert *of* after "value"; line 6, for "α" read ∞ ; in problem 79, where ε occurs insert e.

Page 187, problems 64 and 65 should be 66 and 67.

Page 201, line 12 from bottom, "10 chains" should be 10 rods.

Page 203, line 21, for "$1/m^2.(m^2)!$" read $3/m^2.(m^2)!$; line 22, insert $=$ before $\pi/6$.

Page 204, in lines 1, 2, 3, 4, and 5, insert the sign $=$ before the terms containing π^2 in the numerators.

Page 205, last line of Solution II., for "$a(a+b)$" read $x^2=-a(a+b)$.

Page 206, line 1, for "$\measuredangle BCD$" read $\measuredangle BDC$; for denominator of $\tan^{-1}$ read $c^2+a^2+b^2$.

Page 214, last line, for "36^2" read 35^2.

Page 215, line 6, for "p^2" read p.

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